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PROPAGATION OF WAVES IN AN INHOMOGENEOUS VISCOELASTIC MEDIUM WITH INITIAL STRESSES†

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A closed system of governing equations for the dynamic and geometrical quantities in an inhomogeneous viscoelastic medium with initial stresses is constructed within the framework of the spatial linearized theory of elasticity [1] using the theory of fractures [2]. The geometrical characteristics of the wave front and the paths in an unbounded medium with initial stresses are obtained from the principle of the Fermat functional [3].

The propagation of waves was previously considered in [4, 5] and a calculation was given of the intensity of the wave fronts in a linear inhomogeneous viscoelastic medium with a continuous change in the parameters of the medium which depend on the spatial coordinates without taking account of initial stresses.

1. We will write the relationship between the stress and deformation tensors for an inhomogeneous viscoelastic medium in the form [6]

$$\sigma_{im} = \lambda_1 e_{kk}\delta_{im} + 2\mu_1 e_{im} \tag{1.1}$$

where λ_1 and μ_1 are linear integral operators, the kernels of which depend continuously on the spatial coordinates

$$\begin{aligned} &\lambda_1 = \lambda(1+\Lambda), \quad \Lambda e = \int_0^\infty \Lambda(t', x_i)\, e(t-t')dt' \\ &\mu_1 = \mu(1+\mathrm{M}), \quad \mathrm{M}e = \int_0^\infty \mathrm{M}(t', x_i)\, e(t-t')dt' \end{aligned} \tag{1.2}$$

Relationships (1.1) and (1.2), together with the equations of motion written in a linearized form in Euler coordinates [1]

$$\sigma_{im,i} - \left(u_{m,n}\sigma_{in}^0\right)_{,i} = \rho\ddot{u}_m \quad (m \neq n) \tag{1.3}$$

and the Cauchy formulae

$$2e_{ij} = u_{i,j} + u_{j,i} \tag{1.4}$$

represent a closed system for describing the process of dynamic deformation of an inhomogeneous infinitely viscoelastic medium with initial stresses.

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In formulae (1.2)–(1.4), $\rho(x_i)$, $\lambda(x_i)$, $\mu(x_i)$ are functions of the spatial coordinates, σ^0_{in} are the components of the initial stress tensor, ρ is the density of the medium in the unbounded, unstressed state and u_i are the components of the displacement vector.

Here and henceforth, it is assumed that summation over repeated Latin indices is from one to three and, over repeated Greek indices, from one to two.

The wave of a discontinuity in the stresses in an inhomogeneous viscoelastic system with initial stresses is determined by the isolated surface $\Sigma(t)$, $G(x_i)$ in which the displacements and parameters of the medium are continuous while the stresses, the rates of displacement and the initial stresses undergo a discontinuity.

When account is taken of the initial stresses, the dynamic relationships

$$[\sigma_{ij}]\nu_i = \rho G[\upsilon_j] + [u_{j,n}\sigma^0_{in}]\nu_i \tag{1.5}$$

must be satisfied on the wave surface $\Sigma(t)$.

Here, ν_i are the components of the unit vector normal to $\Sigma(t)$, $G(x_i)$ is the normal velocity of motion of the surface $\Sigma(t)$ in the medium under consideration when account is taken of the initial stresses, 182 denotes the discontinuity of the function f (f^+ is the value of the function on the front side and f^- is the value of the function on the back side of the surface $\Sigma(t)$), where $f^- = 0$ on account of the fact that the part of the medium which is adjacent to the back side of the surface $\Sigma(t)$ is at rest and there are no deformations in it (an unloading wave, that is, in this case the expression in the square brackets reduces to the value of the quantity being considered on the front side of the surface).

It follows from the rheological relationships (1.1), recorded at the discontinuities and the dynamic and kinematic conditions for first-order compatibility [2]

$$[u_{i,j}] = [\upsilon_i]\nu_j / G, \quad [\upsilon_i] = [\partial u_i / \partial t] \tag{1.6}$$

that longitudinal and transverse waves exist in the medium being considered for which

$$\omega_i^{(p)} = \omega^{(p)}\nu_i, \quad \omega_i^{(t)}\nu_i = 0 \tag{1.7}$$

where $\omega_i^{(l)}$ ($l = p, t$) are the components of a vector of amplitude $[\upsilon_i]$. The local velocities of propagation of these waves are respectively

$$G_p^2 = c_p^2(1-\lambda_p^2), \quad G_t^2 = c_t^2(1-\lambda_t^2) \tag{1.8}$$

$$\lambda_p^2 = [\sigma^0_{in}]\nu_i\nu_n / \Lambda_p, \quad \lambda_t^2 = [\sigma^0_{in}]\nu_i\nu_n / \Lambda_t$$

$$\Lambda_p = \lambda + 2\mu, \quad \Lambda_t = \mu, \quad \rho c_p^2 = \Lambda_p, \quad \rho c_t^2 = \Lambda_t$$

Here c_p and c_t are the velocities of the longitudinal and transverse waves in an inhomogeneous viscoelastic medium without initial stresses.

On differentiating relationships (1.1) with respect to t and taking their difference on the different sides of the surface of discontinuity, we find

$$[\dot{\sigma}_{ij}] = \lambda[\dot{e}_{kk}]\delta_{ij} + 2\mu[\dot{e}_{ij}] - \lambda\Lambda(0, x_i)[e_{kk}]\delta_{ij} - 2\mu M(0, x_i)[e_{ij}] \tag{1.9}$$

Let us write the equations of motion (1.3) in terms of the discontinuities

$$[\sigma_{ij,i}] - [u_{j,ni}\sigma^0_{in}] - [u_{j,n}\sigma^0_{in,i}] = \left[\rho \frac{d\upsilon_j}{dt}\right] \tag{1.10}$$

Taking account of the first-order compatibility conditions, the Cauchy formulae (1.4) and the condition

$$[u_{j,ni}] = G^{-3}([\upsilon_j]\delta G / \delta t - G\delta[\upsilon_j] / \delta t - G^2 L_j)\nu_i\nu_n \tag{1.11}$$

we can write expressions (1.9) and (1.10) as follows:

$$\delta[\sigma_{ij}]/\delta t - M_{ij}G = \lambda\delta_{ij}\left(L_k\nu_k + g^{\alpha\beta}[v_k]_{,\alpha}x_{k,\beta}\right) + \mu\left(L_i\nu_j + L_j\nu_i\right) +$$
$$+\mu g^{\alpha\beta}\left([v_i]_{,\alpha}x_{j,\beta} + [v_j]_{,\alpha}x_{i,\beta}\right) + \lambda G^{-1}\Lambda(0,x_i)[v_k]\nu_k\delta_{ij} + \mu G^{-1}M(0,x_i)\left([v_i]\nu_j + [v_j]\nu_i\right) \quad (1.12)$$
$$M_{ij}\nu_i + g^{\alpha\beta}[\sigma_{ij}]_{,\alpha}x_{i,\beta} + G^{-1}L_j[\sigma^0_{in}]\nu_i\nu_n + G^{-1}[\sigma^0_{in,i}][v_j]\nu_n +$$
$$+G^{-3}\left(G\delta[v_j]/\delta t - [v_j]\delta G/\delta t\right)[\sigma^0_{in}]\nu_i\nu_n = \rho\left(\delta[v_j]/\delta t - GL_j\right)$$

The quantities M_{ij} and L_i are defined on the surface $\Sigma(t)$ and characterize the jumps in the first derivatives of the stresses and the velocities of the displacements respectively, $g^{\alpha\beta}$ are the contravariant components of the first fundamental quadratic form, $x_{i,\beta}$ are the derivatives of the Cartesian coordinates x_i with respect to the curvilinear coordinates y_β of the surface ($\beta = 1,2$) and $\delta/\delta t - \delta$ is the derivative with respect to time [2].

In order to eliminate the quantities M_{ij} from Eqs (1.12), we multiply the first of them by ν_i, the second by G, add the results and then use the dynamical relationships (1.5). We obtain

$$(\lambda+\mu)L_k\nu_k\nu_m + \left(\mu - \rho G^2\right)L_m + 2\rho G\delta\omega_m/\delta t - 2(\lambda+\mu)\Omega\omega_k\nu_k\nu_m + (\lambda+\mu)g^{\alpha\beta}x_{m,\beta}(\omega_k\nu_k)_{,\alpha} -$$
$$-2\mu\Omega\omega_m + \rho(\delta G/\delta t)\omega_m + g^{\alpha\beta}\left(\lambda_{,\alpha}x_{m,\beta}\omega_k\nu_k + \mu_{,\alpha}x_{i,\beta}\omega_i\nu_m\right) + G^{-1}\left(\delta[\sigma^0_{in}]/\delta t\right)\omega_m\nu_i\nu_n +$$
$$+G^{-1}[\sigma^0_{in}]\left((\delta\nu_n/\delta t)\omega_m\nu_i - G_{,\alpha}g^{\alpha\beta}x_{i,\beta}\omega_m\nu_n\right) - L_m[\sigma^0_{kn}]\nu_k\nu_n -$$
$$-[\sigma^0_{in,i}]\omega_m\nu_n + G^{-1}\left(\lambda\Lambda(0,x_i)\omega_k\nu_k\nu_m + \mu M(0,x_i)(\omega_m + \omega_k\nu_k\nu_m)\right) = 0 \quad (1.13)$$

The relationships

$$\nu_{k,\alpha} = -g^{\sigma\tau}b_{\sigma\alpha}x_{k,\tau}, \quad g^{\alpha\beta}g^{\sigma\tau}b_{\alpha\tau}x_{j,\tau}x_{j,\beta} = 2\Omega$$
$$x_{k,\alpha}\nu_k = 0, \quad \nu_i\delta\nu_i/\delta t = 0$$

have been taken into account in the derivation, where Ω is the mean curvature of the wave surface and $b_{\sigma\alpha}$ are the coefficients of the second quadratic form of the surface $\Sigma(t)$.

On multiplying (1.1) by ν_m, summing over the repeated index and then taking account of the first formulae of (1.7) and (1.8), we obtain the differential equation for the longitudinal wave. For the transverse wave, for which $\omega_i^{(t)}\nu_i = 0$, we transform relationship (1.13) taking account of the second formula of (1.8). On subsequently changing to the variable $s \geqslant 0$, which denotes the distance along the normals to the surface $\Sigma(t_0)$, we obtain the equations for the change in the amplitude of the longitudinal and transverse waves during their propagation

$$\frac{d\omega^{(l)}}{ds} = \left\{\Omega_l - \frac{1}{2}\frac{d}{ds}\left(c_l\left(1-\lambda_l^2\right)^{1/2}\right) + \frac{[\sigma^0_{in}]\nu_i\nu_n}{\rho c_l^2\left(1-\lambda_l^2\right)}\Omega_l - \frac{B_l}{2\rho c_l^3\left(1-\lambda_l^2\right)^{3/2}}\right\}\omega^{(l)} \quad (l = p,t) \quad (1.14)$$
$$B_p = \lambda\Lambda(0,s) + 2\mu M(0,s), \quad B_t = \mu M(0,s)$$

Equations (1.14) contain the geometric invariant Ω_l, that is, the mean curvature of the wave front as the unknown function and, consequently, they are not closed. According to results which have been previously obtained [7, 8] and formulae (1.8), we obtain the equation for Ω_l in the radial system of coordinates

$$d\Omega_l/ds = 2\Omega_l^2 - K_l - \tfrac{1}{2}(Q_1 - Q_2 - R_1)$$
$$Q_1 = c_l^{-1}g^{\alpha\beta}\left(1-\lambda_l^2\right)^{-1}c_{l,\alpha\beta}, \quad Q_2 = g^{\alpha\beta}\lambda_l\left(c_l\left(1-\lambda_l^2\right)\right)^{-1}(\lambda_l c_l)_{,\alpha\beta} \quad (1.15)$$

$$R_1 = g^{\alpha\beta}\lambda_l^2\left(1-\lambda_l^2\right)^{-2}\left(\ln\lambda_l\right)_{,\alpha}\left(\ln\lambda_l\right)_{,\beta}$$

where K_l is the Gaussian curvature of the surface $\Sigma(t)$ which is determined from the equation

$$dK_l / ds = 2\Omega_l K_l + 2\Omega_l\left(Q_1 - Q_2 - R_1\right) - P_1 + P_2 + R_2$$

$$P_1 = c_l^{-1} b^{\alpha\beta} c_{l,\alpha\beta}(1-\lambda_l^2)^{-1}, \quad P_2 = b^{\alpha\beta}\lambda_l(c_l(1-\lambda_l^2))^{-1}(\lambda_l c_l)_{,\alpha\beta} \tag{1.16}$$

$$R_2 = b^{\alpha\beta}\lambda_l^2\left(1-\lambda_l^2\right)^{-2}\left(\ln\lambda_l\right)_{,\alpha}\left(\ln\lambda_l\right)_{,\beta}$$

We find the equations for the trajectory of a ray from the Fermat functional principle, taking account of relationships (1.8)

$$dv_i / ds = -g^{\alpha\beta}\chi_{l,\alpha} x_{i,\beta}, \quad v_i = dx_i / ds$$

$$dx_{i,\alpha} / ds = \chi_{l,\alpha} v_i - g^{\delta\gamma} b_{\delta\alpha} x_{i,\gamma}, \quad \chi_l = \ln\left(c_l\left(1-\lambda_l^2\right)^{1/2}\right)$$

The covariant and contravariant components of the first and second quadratic forms of the surface, taking into account the initial stresses, satisfy equations [2, 8]

$$db_{\alpha\beta} / ds = \chi_{l,\alpha\beta} + \chi_{l,\alpha}\chi_{l,\beta} - g^{\eta\delta} b_{\alpha\eta} b_{\beta\delta}$$

$$db^{\alpha\beta} / ds = g^{\alpha\eta} g^{\beta\delta}\left(\chi_{l,\eta\delta} + \chi_{l,\eta}\chi_{l,\delta}\right) + 3 g_{\eta\delta} b^{\alpha\eta} b^{\beta\delta}$$

$$dg_{\alpha\beta} / ds = -2 b_{\alpha\beta}, \quad dg^{\alpha\beta} / ds = 2 b^{\alpha\beta}$$

For specified functions c_l, λ_l and initial values ω_0, Ω_0, K_0, $b^0_{\alpha\beta}$, $b_0^{\alpha\beta}$, $g^0_{\alpha\beta}$, $g_0^{\alpha\beta}$ the system has a unique solution. On eliminating the Gaussian curvature K from Eqs (1.15) and (1.16), we obtain

$$\ddot{\Omega} - 6\Omega\dot{\Omega} + 4\dot{\Omega}^3 = -3\Omega\left(Q_1 - Q_2 - R_1\right) + \tfrac{1}{2}\left(\dot{Q}_1 - \dot{Q}_2 - \dot{R}_1\right) + P_1 - P_2 - R_2 \quad \left(\dot{\Omega} = d\Omega / ds\right) \tag{1.17}$$

On solving Eqs (1.16) and (1.17) by the method of successive approximations when $n = 0, 1, 2, \ldots$, we obtain that the zeroth approximation corresponds to a homogeneous medium [2]. The solution of the equations of the first approximation with null initial conditions will be $\Omega^{(1)} = K^{(1)} = 0$. The solutions of the equations for $\Omega^{(2)}$ and $K^{(2)}$ are quite complicated and we shall therefore consider the case when $\Omega_0 = K_0 = 0$, that is, the wave is planar at the beginning. In this case $\Omega^{(0)} = K^{(0)} = \Omega^{(1)} = K^{(1)} = 0$, and, in the second approximation, we obtain

$$\Omega = \Omega^{(2)} = \frac{1}{2}\int_0^s\left(Q_1(s) - Q_2(s) - R_1(s)\right) ds - \int_0^s K^{(2)}(s) ds \tag{1.18}$$

$$K = K^{(2)} = -\int_0^s\left(P_1(s) - P_2(s) - R_2(s)\right) ds$$

Let us now determine the level of the amplitude ω which satisfies Eq. (1.14). In order to do this, we select the magnitude of the gradient $G_l = c_l(1-\lambda_l^2)^{1/2}$ as the parameter which determines the order of the approximations. Then, $d\chi_l/ds$ is of the first order and $\chi_{l,\alpha}$ and $\chi_{l,\beta}$ are of the second order. A homogeneous medium corresponds to the zeroth order. In the first approximation, account is taken of the rate of change of the inhomogeneity of a viscoelastic medium with initial stresses along a ray while, in the second approximation, account is taken of the rate of change of the inhomogeneity of a viscoelastic medium with initial stresses at right angles to the ray.

On substituting $\omega = \omega^{(0)} + \omega^{(1)} + \ldots$ into (1.14) and solving it by the method of approximations, we obtain

$$d\omega^{(0)} / ds = \Omega^{(0)}(s)\omega^{(0)}, \quad d\omega^{(1)} / ds = \Omega^{(0)}\omega^{(1)} + \left(\Omega^{(1)} - f^{(1)}\right)\omega^{(0)}$$

$$d\omega^{(2)}/ds = \Omega^{(0)}\omega^{(2)} + \left(\Omega^{(1)} - f^{(1)}\right)\omega^{(1)} + \left(\Omega^{(2)} - g^{(1)}\Omega^{(1)}\right)\omega^{(0)} \tag{1.19}$$

$$f^{(1)} = \tfrac{1}{2}d\chi_i/ds + g_i^{(1)}\Omega^{(0)} + B_i\left(2\rho c_i^3\left(1-\lambda_i^2\right)^{3/2}\right)^{-1}$$

$$g_i^{(1)} = -\left(\rho c_i^2\left(1-\lambda_i^2\right)\right)^{-1}\left[\sigma_{in}^0\right]\nu_i\nu_n$$

In the case of Eqs (1.19), we impose the initial conditions

$$\omega^{(0)}(0) = \omega_0^{(0)}, \quad \omega^{(i)}(0) = 0 \quad (i = 1,2,\ldots) \tag{1.20}$$

The solution of Eqs (1.19), taking account of (1.20) and the fact that $\Omega^{(i)}(0)=0$ can be written in the form

$$\omega^{(0)} = \frac{\omega_0^{(0)}}{\sqrt{\Psi}}, \quad \omega^{(1)} = -\frac{\omega_0^{(0)}}{\sqrt{\Psi}}\int_0^s f^{(1)}(s_1)ds_1$$

$$\omega^{(2)} = \frac{\omega_0^{(0)}}{\sqrt{\Psi}}\left\{\int_0^s\int_0^s f^{(1)}(s_1)f^{(1)}(s_2)ds_1ds_2 - \int_0^s \Omega^{(2)}(s_2)ds_2\right\} \tag{1.21}$$

$$\Psi = 1 - 2\Omega_0 s + K_0 s^2$$

2. Let us now consider a layered viscoelastic medium with initial stresses which is characterized by elastic moduli $\lambda(x)$, $\mu(x)$, density $\rho(x)$, relaxation kernels $\Lambda(0, x)$, $M(0, x)$ and initial stresses σ_{ij}^0.

At the instant of time $t = 0$, delamination of the layers occurs in the y, z plane. The unloading wave propagates along the x-axis. From (1.18), the velocity of propagation of the wave will then have the form

$$G_p(x) = c_p(x)\left(1-\lambda_p^2(x)\right)^{1/2}, \quad \lambda_p^2 = \left[\sigma_{xx}^0\right]/(\lambda+2\mu) \tag{2.1}$$

Since $g_{\alpha\beta} = b_{\alpha\beta} = 0$ and when $x = 0$, $\Omega_0 = K_0 = 0$, we obtain from Eqs (1.15) and (1.16) that $\Omega = K = 0$, while, from Eq. (1.14), we find the dependence of the level of the intensity of the wave on the velocity, the relaxation kernels and the initial stresses

$$\omega^{(p)} = \omega_0^{(p)}\left(c_p\left(1-\lambda_p^2\right)^{1/2}\right)^{-1/2}\exp\left\{-\frac{1}{2}\int_0^x \frac{\lambda\Lambda(0,x)+2\mu M(0,x)}{\rho c_p^3\left(1-\lambda_p^2\right)^{3/2}}dx\right\} \tag{2.2}$$

where $\omega_0^{(p)}$ is the value of the function $\omega^{(p)}$ when $x = 0$.

On specifying the form of $c_p(x)$, $\Delta(0, x)$, $M(0, x)$ and $[\sigma_{xx}^0]$, in (2.2), we obtain the nature of the change in the level of the intensity of the wave in an inhomogeneous viscoelastic medium with initial stresses.

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